



Corrigendum

Corrigendum to “Deformation quantization in algebraic geometry” [Adv. Math. 198 (2005) 383–432]

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0. Introduction

This note contains a correction of the *proofs* of the main results of [3], namely Theorems 0.1 and 0.2. The results are correct as originally stated.

The mistake in my original proofs was discovered Michel Van den Bergh, and I thank him for calling my attention to it. The way to fix the proofs is essentially contained in his paper [2].

Let me begin by explaining the mistake. As can be seen in Example 0.1 below, the mistake itself is of a rather elementary nature, but it was obscured by the complicated context.

Suppose \mathbb{K} is a field of characteristic 0, and X is a smooth separated n -dimensional scheme over \mathbb{K} . Recall that the coordinate bundle $\text{Coor } X$ is an infinite dimensional bundle over X , with free action by the group $\text{GL}_{n,\mathbb{K}}$. The quotient bundle is by definition

$$\text{LCC } X := \text{Coor } X / \text{GL}_{n,\mathbb{K}},$$

and the projection $\pi_{\text{gl}} : \text{Coor } X \rightarrow \text{LCC } X$ is a $\text{GL}_{n,\mathbb{K}}$ -torsor.

The erroneous (implicit) assertion in [3] is that the de Rham complexes satisfy

$$(\pi_{\text{gl}*} \Omega_{\text{Coor } X})^{\text{GL}_n(\mathbb{K})} = \Omega_{\text{LCC } X}.$$

From that I deduced (incorrectly, top of page 424) that the Maurer–Cartan form ω_{MC} is a global section of the sheaf

$$\Omega_{\text{LCC } X}^1 \widehat{\otimes}_{\mathcal{O}_{\text{LCC } X}} \pi_{\text{gl}}^* (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly}, X}^0).$$

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(This is false, as can be seen from [2, Lemma 6.5.1].) This led to many incorrect formulas in [3, Section 7].

The correct thing to do is to work with the infinitesimal action of the Lie algebra $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{K})$. For $v \in \mathfrak{g}$ one has the contraction (inner derivative) ι_v , which is a degree -1 derivation of the de Rham complex $\pi_{\mathfrak{gl}*} \Omega_{\text{Coor } X}$. Recall that the Lie derivative is $L_v := d \circ \iota_v + \iota_v \circ d$. A local section $\omega \in \pi_{\mathfrak{gl}*} \Omega_{\text{Coor } X}$ is said to be \mathfrak{g} -invariant if $\iota_v(\omega) = L_v(\omega) = 0$ for all $v \in \mathfrak{g}$. According to [2, Lemma 9.2.3] one has

$$(\pi_{\mathfrak{gl}*} \Omega_{\text{Coor } X})^{\mathfrak{g}} = \Omega_{\text{LCC } X}.$$

It is worthwhile to note that in my incorrect proof there was no need to invoke Kontsevich's property (P5) from [1]. The correct proof does require property (P5)—cf. [2, Lemma 9.2.1].

Example 0.1. Here is a simplified example. Suppose G is the affine algebraic group $\text{GL}_{1,\mathbb{K}} = \text{Spec } \mathbb{K}[t, t^{-1}]$, and X is the variety G , with regular left action. The group of rational points is $G(\mathbb{K}) = \mathbb{K}^{\times}$. The action of G on X is free, the invariant ring is $\mathcal{O}(X)^{G(\mathbb{K})} = \mathbb{K}$, and the quotient is $X/G = \text{Spec } \mathbb{K}$. For the de Rham complex

$$\Omega(X) = \mathcal{O}(X) \oplus \Omega^1(X) = \mathbb{K}[t, t^{-1}] \oplus \mathbb{K}[t, t^{-1}] \cdot dt$$

we have $\Omega(X)^{G(\mathbb{K})} \neq \mathbb{K}$, since it contains $t^{-1} dt$. But for the infinitesimal action of the Lie algebra $\mathfrak{g} := \mathfrak{gl}_1(\mathbb{K})$ it is easy to see that $\Omega(X)^{\mathfrak{g}} = \mathbb{K}$.

After some deliberation I decided that the best way to present the correction is by completely rewriting [3, Section 7]. This is Section 1 below. Section 2 contains some additional minor corrections to [3].

1. The global L_{∞} quasi-isomorphism

This is a revised version of [3, Section 7]. In this section we prove the main results of the paper [3], namely Theorem 0.1 (which is repeated here as Corollary 1.19) and Theorem 0.2 (which is repeated here, with more details, as Theorem 1.2). Throughout \mathbb{K} is a field containing \mathbb{R} , and X is a smooth irreducible separated n -dimensional scheme over \mathbb{K} . We use all notation, definitions and results of [3, Sections 1–6] freely. However, the bibliography references relate to the list at the end of this note.

Suppose $U = \{U_0, \dots, U_m\}$ is an open covering of the scheme X , consisting of affine open sets, each admitting an étale coordinate system, namely an étale morphism $U_i \rightarrow \mathbb{A}_{\mathbb{K}}^n$. For every i let $\sigma_i : U_i \rightarrow \text{LCC } X$ be the corresponding section of $\pi_{\text{LCC}} : \text{LCC } X \rightarrow X$, and let σ be the resulting simplicial section (see [3, Theorem 6.5]).

Let \mathcal{M} be a bounded below complex of quasi-coherent \mathcal{O}_X -modules. The mixed resolution $\text{Mix}_U(\mathcal{M})$ was defined in [3, Section 6]. For any integer i let

$$G^i \text{Mix}_U(\mathcal{M}) := \bigoplus_{j=i}^{\infty} \text{Mix}_U^j(\mathcal{M}),$$

so $\{G^i \operatorname{Mix}_U(\mathcal{M})\}_{i \in \mathbb{Z}}$ is a descending filtration of $\operatorname{Mix}_U(\mathcal{M})$ by subcomplexes, with $G^i \operatorname{Mix}_U(\mathcal{M}) = \operatorname{Mix}_U(\mathcal{M})$ for $i \leq 0$, and $\bigcap_i G^i \operatorname{Mix}_U(\mathcal{M}) = 0$. Let

$$\operatorname{gr}_G^i \operatorname{Mix}_U(\mathcal{M}) := G^i \operatorname{Mix}_U(\mathcal{M}) / G^{i+1} \operatorname{Mix}_U(\mathcal{M})$$

and $\operatorname{gr}_G \operatorname{Mix}_U(\mathcal{M}) := \bigoplus_i \operatorname{gr}_G^i \operatorname{Mix}_U(\mathcal{M})$.

By [3, Proposition 6.3], if \mathcal{G}_X is either $\mathcal{T}_{\text{poly}, X}$ or $\mathcal{D}_{\text{poly}, X}$, then $\operatorname{Mix}_U(\mathcal{G}_X)$ is a sheaf of DG Lie algebras on X , and the inclusion

$$\eta_{\mathcal{G}} : \mathcal{G}_X \rightarrow \operatorname{Mix}_U(\mathcal{G}_X)$$

is a DG Lie algebra quasi-isomorphism.

Note that if $\phi : \operatorname{Mix}_U(\mathcal{M}) \rightarrow \operatorname{Mix}_U(\mathcal{N})$ is a homomorphism of complexes that respects the filtration $\{G^i \operatorname{Mix}_U\}$, then there exists an induced homomorphism of complexes

$$\operatorname{gr}_G(\phi) : \operatorname{gr}_G \operatorname{Mix}_U(\mathcal{M}) \rightarrow \operatorname{gr}_G \operatorname{Mix}_U(\mathcal{N}).$$

Suppose \mathcal{G} and \mathcal{H} are sheaves of DG Lie algebras on a topological space Y . An L_∞ morphism $\Psi : \mathcal{G} \rightarrow \mathcal{H}$ is a sequence of sheaf morphisms $\psi_j : \prod^j \mathcal{G} \rightarrow \mathcal{H}$, such that for every open set $V \subset Y$ the sequence $\{\Gamma(V, \psi_j)\}_{j \geq 1}$ is an L_∞ morphism $\Gamma(V, \mathcal{G}) \rightarrow \Gamma(V, \mathcal{H})$. If $\psi_1 : \mathcal{G} \rightarrow \mathcal{H}$ is a quasi-isomorphism then Ψ is called an L_∞ quasi-isomorphism.

Recall that there is a canonical quasi-isomorphism of complexes of \mathcal{O}_X -modules

$$\mathcal{U}_1 : \mathcal{T}_{\text{poly}, X} \rightarrow \mathcal{D}_{\text{poly}, X}. \quad (1.1)$$

According to [4, Theorem 4.17], the induced homomorphism

$$\operatorname{gr}_G(\operatorname{Mix}_U(\mathcal{U}_1)) : \operatorname{gr}_G \operatorname{Mix}_U(\mathcal{T}_{\text{poly}, X}) \rightarrow \operatorname{gr}_G \operatorname{Mix}_U(\mathcal{D}_{\text{poly}, X})$$

is a quasi-isomorphism.

Theorem 1.2. *Let X be an irreducible smooth separated \mathbb{K} -scheme. Let $U = \{U_0, \dots, U_m\}$ be an open covering of X consisting of affine open sets, each admitting an étale coordinate system, and let σ be the associated simplicial section of the bundle $\operatorname{LCC} X \rightarrow X$. Then there is an induced L_∞ quasi-isomorphism*

$$\Psi_\sigma = \{\Psi_{\sigma; j}\}_{j \geq 1} : \operatorname{Mix}_U(\mathcal{T}_{\text{poly}, X}) \rightarrow \operatorname{Mix}_U(\mathcal{D}_{\text{poly}, X}).$$

The homomorphism $\Psi_{\sigma; 1}$ respects the filtration $\{G^i \operatorname{Mix}_U\}$, and

$$\operatorname{gr}_G(\Psi_{\sigma; 1}) = \operatorname{gr}_G(\operatorname{Mix}_U(\mathcal{U}_1)).$$

Proof. Let Y be some \mathbb{K} -scheme, and denote by \mathbb{K}_Y the constant sheaf. For any p we view Ω_Y^p as a discrete inv \mathbb{K}_Y -module, and we put on $\Omega_Y = \bigoplus_{p \in \mathbb{N}} \Omega_Y^p$ the direct sum dir-inv structure. So Ω_Y is a discrete (and hence complete) DG algebra in $\operatorname{Dir Inv Mod} \mathbb{K}_Y$.

We shall abbreviate $\mathcal{A} := \Omega_{\operatorname{Coor} X}$, so that $\mathcal{A}^0 = \mathcal{O}_{\operatorname{Coor} X}$ etc. As explained above, \mathcal{A} is a DG algebra in $\operatorname{Dir Inv Mod} \mathbb{K}_{\operatorname{Coor} X}$, with discrete (but not trivial) dir-inv module structure.

There are sheaves of DG Lie algebras $\mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!])$ and $\mathcal{A} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[\![t]\!])$ on the scheme $\text{Coor } X$. The differentials are $d_{\text{for}} = d \otimes \mathbf{1}$ and $d_{\text{for}} + \mathbf{1} \otimes d_{\mathcal{D}}$ respectively. As explained just prior to [3, Theorem 3.16], \mathcal{U} extends to a continuous \mathcal{A} -multilinear L_{∞} morphism

$$\mathcal{U}_{\mathcal{A}} = \{\mathcal{U}_{\mathcal{A};j}\}_{j \geq 1} : \mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!]) \rightarrow \mathcal{A} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[\![t]\!])$$

of sheaves of DG Lie algebras on $\text{Coor } X$.

The MC form $\omega := \omega_{\text{MC}}$ is a global section of $\mathcal{A}^1 \widehat{\otimes} \mathcal{T}_{\text{poly}}^0(\mathbb{K}[\![t]\!])$ satisfying the MC equation in the DG Lie algebra $\mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!])$. See [3, Proposition 5.9]. According to [3, Theorem 3.16], the global section $\omega' := \mathcal{U}_{\mathcal{A};1}(\omega) \in \mathcal{A}^1 \widehat{\otimes} \mathcal{D}_{\text{poly}}^0(\mathbb{K}[\![t]\!])$ is a solution of the MC equation in the DG Lie algebra $\mathcal{A} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[\![t]\!])$, and there is a continuous \mathcal{A} -multilinear L_{∞} morphism

$$\mathcal{U}_{\mathcal{A},\omega} = \{\mathcal{U}_{\mathcal{A},\omega;j}\}_{j \geq 1} : (\mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!]))_{\omega} \rightarrow (\mathcal{A} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[\![t]\!]))_{\omega'}$$

between the twisted DG Lie algebras. The formula is

$$\mathcal{U}_{\mathcal{A},\omega;j}(\gamma_1 \cdots \gamma_j) = \sum_{k \geq 0} \frac{1}{(j+k)!} \mathcal{U}_{\mathcal{A};j+k}(\omega^k \cdot \gamma_1 \cdots \gamma_j) \quad (1.3)$$

for $\gamma_1, \dots, \gamma_j \in \mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!])$. The two twisted DG Lie algebras have differentials $d_{\text{for}} + \text{ad}(\omega)$ and $d_{\text{for}} + \text{ad}(\omega') + \mathbf{1} \otimes d_{\mathcal{D}}$ respectively.

This sum in (1.3) is actually finite, the number of nonzero terms in it depending on the bidegree of $\gamma_1 \cdots \gamma_j$. Indeed, if $\gamma_1 \cdots \gamma_j \in \mathcal{A}^q \widehat{\otimes} \mathcal{T}_{\text{poly}}^p(\mathbb{K}[\![t]\!])$, then

$$\mathcal{U}_{\mathcal{A};j+k}(\omega^k \cdot \gamma_1 \cdots \gamma_j) \in \mathcal{A}^{q+k} \widehat{\otimes} \mathcal{D}_{\text{poly}}^{p+1-j-k}(\mathbb{K}[\![t]\!]), \quad (1.4)$$

which is zero for $k > p - j + 2$; see proof of [4, Theorem 3.23].

By [3, Theorem 5.6] (the universal Taylor expansions) there are canonical isomorphisms of graded Lie algebras in $\text{Dir Inv Mod } \mathbb{K}_{\text{Coor } X}$

$$\mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!]) \cong \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \pi_{\text{coor}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly},X}) \quad (1.5)$$

and

$$\mathcal{A} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[\![t]\!]) \cong \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \pi_{\text{coor}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X}). \quad (1.6)$$

[3, Proposition 5.8] tells us that

$$d_{\text{for}} + \text{ad}(\omega) = \nabla_{\mathcal{P}}$$

under these identifications. Therefore

$$\begin{aligned} \mathcal{U}_{\mathcal{A},\omega} &= \{\mathcal{U}_{\mathcal{A},\omega;j}\}_{j \geq 1} : \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \pi_{\text{coor}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly},X}) \\ &\rightarrow \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \pi_{\text{coor}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X}) \end{aligned} \quad (1.7)$$

is a continuous \mathcal{A} -multilinear L_∞ morphism between these DG Lie algebras, whose differentials are $\nabla_{\mathcal{P}}$ and $\nabla_{\mathcal{P}} + \mathbf{1} \otimes d_{\mathcal{P}}$ respectively. As in the proof of [3, Theorem 5.6], under the identifications (1.5) and (1.6) we have the equality

$$\mathcal{U}_{\mathcal{A};1} = \mathbf{1} \otimes \pi_{\text{coor}}^*(\mathbf{1} \otimes \mathcal{U}_1); \quad (1.8)$$

i.e., it is the pullback of the map (1.1).

Let us filter the DG algebra \mathcal{A} by the descending filtration $\{G^i \mathcal{A}\}_{i \in \mathbb{Z}}$, where $G^i \mathcal{A} := \bigoplus_{j=i}^{\infty} \mathcal{A}^j$. The DG Lie algebras appearing in Eq. (1.7) inherit this filtration. From formulas (1.3) and (1.4) we see that the homomorphism of complexes $\mathcal{U}_{\mathcal{A},\omega;1}$ respects the filtration, and from (1.8) we see that

$$\text{gr}_G(\mathcal{U}_{\mathcal{A},\omega;1}) = \text{gr}_G(\mathcal{U}_{\mathcal{A};1}) = \mathbf{1} \otimes \pi_{\text{coor}}^*(\mathbf{1} \otimes \mathcal{U}_1).$$

Let $n := \dim X$. As noted earlier, the action of $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{K})$ gives

$$(\pi_{\text{gl}*} \mathcal{A})^{\mathfrak{g}} = (\pi_{\text{gl}*} \Omega_{\text{Coor } X})^{\mathfrak{g}} = \Omega_{\text{LCC } X}.$$

According to [2, Lemma 9.2.1], the L_∞ morphism $\mathcal{U}_{\mathcal{A},\omega}$ commutes with the action of the Lie algebra \mathfrak{g} . Therefore $\mathcal{U}_{\mathcal{A},\omega}$ descends (i.e. restricts) to a continuous $\Omega_{\text{LCC } X}$ -multilinear L_∞ morphism

$$\begin{aligned} \mathcal{U}_{\mathcal{A},\omega}^{\mathfrak{g}} : \Omega_{\text{LCC } X} \widehat{\otimes}_{\Omega_{\text{LCC } X}} \pi_{\text{lcc}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly},X}) \\ \rightarrow \Omega_{\text{LCC } X} \widehat{\otimes}_{\Omega_{\text{LCC } X}} \pi_{\text{lcc}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X}). \end{aligned} \quad (1.9)$$

The DG Lie algebras in formula (1.9) also have filtrations $\{G^j\}_{j \in \mathbb{Z}}$, the homomorphism $\mathcal{U}_{\mathcal{A},\omega;1}^{\mathfrak{g}}$ respects this filtration, and we now have

$$\text{gr}_G(\mathcal{U}_{\mathcal{A},\omega;1}^{\mathfrak{g}}) = \text{gr}_G(\mathcal{U}_{\mathcal{A};1}^{\mathfrak{g}}) = \mathbf{1} \otimes \pi_{\text{lcc}}^*(\mathbf{1} \otimes \mathcal{U}_1). \quad (1.10)$$

According to [3, Theorem 6.4] there are induced operators

$$\psi_{\sigma;j} := \sigma^*(\mathcal{U}_{\mathcal{A},\omega;j}^{\mathfrak{g}}) : \prod_{j=1}^j \text{Mix}_U(\mathcal{T}_{\text{poly},X}) \rightarrow \text{Mix}_U(\mathcal{D}_{\text{poly},X})$$

for $j \geq 1$. The L_∞ identities in [3, Definition 3.7], when applied to the L_∞ morphism $\mathcal{U}_{\mathcal{A},\omega}^{\mathfrak{g}}$, are of the form considered in [3, Theorem 6.4(iii)]. Therefore these identities are preserved by σ^* , and we conclude that the sequence $\psi_\sigma = \{\psi_{\sigma;j}\}_{j=1}^\infty$ is an L_∞ morphism. Furthermore, $\psi_{\sigma;1}$ respects the filtration $\{G^i \text{Mix}_U\}$, and from (1.10) we get

$$\text{gr}_G(\psi_{\sigma;1}) = \text{gr}_G(\sigma^*(\mathcal{U}_{\mathcal{A};1}^{\mathfrak{g}})) = \text{gr}_G(\text{Mix}_U(\mathcal{U}_1)). \quad (1.11)$$

According to [4, Theorem 4.17] the homomorphism $\text{gr}_G(\text{Mix}_U(\mathcal{U}_1))$ is a quasi-isomorphism. Since the complexes $\text{Mix}_U(\mathcal{T}_{\text{poly},X})$ and $\text{Mix}_U(\mathcal{D}_{\text{poly},X})$ are bounded below, and the filtration is nonnegative and exhaustive, it follows that $\psi_{\sigma;1}$ is also a quasi-isomorphism. \square

Corollary 1.12. *Taking global sections in Theorem 1.2 we get an L_∞ quasi-isomorphism*

$$\Gamma(X, \Psi_\sigma) = \left\{ \Gamma(X, \Psi_{\sigma;j}) \right\}_{j \geq 1} : \Gamma(X, \text{Mix}_U(\mathcal{T}_{\text{poly}, X})) \rightarrow \Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X})).$$

Proof. Theorem 1.2 tells us that $\Psi_{\sigma;1}$ is a quasi-isomorphism of complexes of sheaves. By [3, Theorem 6.2] it follows that

$$\Gamma(X, \Psi_{\sigma;1}) : \Gamma(X, \text{Mix}_U(\mathcal{T}_{\text{poly}, X})) \rightarrow \Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X}))$$

is a quasi-isomorphism. \square

Corollary 1.13. *The data (U, σ) induce a bijection*

$$\text{MC}(\Psi_\sigma) : \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{T}_{\text{poly}, X}))[[\hbar]]^+) \xrightarrow{\cong} \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X}))[[\hbar]]^+).$$

Proof. Use Corollary 1.12 and [3, Corollary 3.10]. \square

Recall that $\mathcal{T}_{\text{poly}}(X) = \Gamma(X, \mathcal{T}_{\text{poly}, X})$ and $\mathcal{D}_{\text{poly}}^{\text{nor}}(X) = \Gamma(X, \mathcal{D}_{\text{poly}, X}^{\text{nor}})$, and the latter is the DG Lie algebra of global poly differential operators that vanish if one of their arguments is 1.

Suppose $f : X' \rightarrow X$ is an étale morphism. According to [4, Proposition 4.6] there are DG Lie algebra homomorphisms $f^* : \mathcal{T}_{\text{poly}}(X) \rightarrow \mathcal{T}_{\text{poly}}(X')$ and $f^* : \mathcal{D}_{\text{poly}}^{\text{nor}}(X) \rightarrow \mathcal{D}_{\text{poly}}^{\text{nor}}(X')$. These homomorphisms extend to formal coefficients, and we get functions

$$\text{MC}(f^*) : \text{MC}(\mathcal{T}_{\text{poly}}(X)[[\hbar]]^+) \rightarrow \text{MC}(\mathcal{T}_{\text{poly}}(X')[[\hbar]]^+)$$

etc.

One says that X is a \mathcal{D} -affine variety if $H^q(X, \mathcal{M}) = 0$ for every quasi-coherent left \mathcal{D}_X -module \mathcal{M} and every $q > 0$.

Theorem 1.14. *Let X be an irreducible smooth separated \mathbb{K} -scheme. Assume X is \mathcal{D} -affine. Then there is a canonical function*

$$Q : \text{MC}(\mathcal{T}_{\text{poly}}(X)[[\hbar]]^+) \rightarrow \text{MC}(\mathcal{D}_{\text{poly}}^{\text{nor}}(X)[[\hbar]]^+)$$

called the quantization map. It has the following properties:

- (i) *The function Q preserves first order terms.*
- (ii) *The function Q respects étale morphisms. Namely if X' is another \mathcal{D} -affine scheme, with quantization map Q' , and if $f : X' \rightarrow X$ is an étale morphism, then*

$$Q' \circ \text{MC}(f^*) = \text{MC}(f^*) \circ Q.$$

- (iii) *If X is affine, then Q is bijective.*

- (iv) *The function Q is characterized as follows. Choose an open covering $U = \{U_0, \dots, U_m\}$ of X consisting of affine open sets, each admitting an étale coordinate system. Let σ be*

the associated simplicial section of the bundle $\text{LCC } X \rightarrow X$. Then there is a commutative diagram

$$\begin{array}{ccc} \text{MC}(\mathcal{T}_{\text{poly}}(X)[\hbar]^+) & \xrightarrow{Q} & \text{MC}(\mathcal{D}_{\text{poly}}^{\text{nor}}(X)[\hbar]^+) \\ \text{MC}(\eta_T) \downarrow & & \downarrow \text{MC}(\eta_D) \\ \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{T}_{\text{poly}, X}))[\hbar]^+) & \xrightarrow{\text{MC}(\Psi_\sigma)} & \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X}))[\hbar]^+) \end{array}$$

in which the arrows $\text{MC}(\Psi_\sigma)$ and $\text{MC}(\eta_D)$ are bijections. Here Ψ_σ is the L_∞ quasi-isomorphism from Theorem 1.2, and η_T, η_D are the inclusions of DG Lie algebras.

Let us elaborate a bit on the statement above. It says that to any MC solution $\alpha = \sum_{j=1}^\infty \alpha_j \hbar^j \in \mathcal{T}_{\text{poly}}^1(X)[\hbar]^+$ there corresponds an MC solution $\beta = \sum_{j=1}^\infty \beta_j \hbar^j \in \mathcal{D}_{\text{poly}}^{\text{nor}, 1}(X)[\hbar]^+$. The element $\beta = Q(\alpha)$ is uniquely determined up to gauge equivalence by the group $\exp(\mathcal{D}_{\text{poly}}^{\text{nor}, 0}(X)[\hbar]^+)$. Given any local sections $f, g \in \mathcal{O}_X$ one has

$$\frac{1}{2}(\beta_1(f, g) - \beta_1(g, f)) = \alpha_1(f, g) \in \mathcal{O}_X. \quad (1.15)$$

The quantization map Q can be calculated (at least in theory) using the collection of sections σ and the universal formulas for deformation in [3, Theorem 3.13].

We will need a lemma before proving the theorem.

Lemma 1.16. *Let $f, g \in \mathcal{O}_X = \mathcal{D}_{\text{poly}, X}^{-1}$ be local sections.*

(1) *For any $\beta \in \text{Mix}_U^0(\mathcal{D}_{\text{poly}, X}^1)$ one has*

$$[[\beta, f], g] = \beta(g, f) - \beta(f, g) \in \text{Mix}_U^0(\mathcal{O}_X).$$

(2) *For any $\beta \in \text{Mix}_U^1(\mathcal{D}_{\text{poly}, X}^0) \oplus \text{Mix}_U^2(\mathcal{D}_{\text{poly}, X}^{-1})$ one has $[[\beta, f], g] = 0$.*

(3) *Let $\gamma \in \text{Mix}_U(\mathcal{D}_{\text{poly}, X})^0$, and define $\beta := (d_{\text{mix}} + d_D)(\gamma)$. Then $[[\beta, f], g] = 0$.*

Proof. (1) [3, Proposition 6.3] implies that the embedding [3, (6.1)]:

$$\begin{aligned} & \text{Mix}_U(\mathcal{D}_{\text{poly}, X}) \\ & \subset \bigoplus_{p, q, r} \prod_{j \in \mathbb{N}} \prod_{i \in \Delta_j^m} g_i * g_i^{-1} (\Omega^q(\Delta_{\mathbb{K}}^j) \widehat{\otimes} (\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly}, X}^r)) \end{aligned}$$

is a DG Lie algebra homomorphism. So by continuity we might as well assume that $\beta = aD$ with $a \in \Omega_X^0 = \mathcal{O}_X$ and $D \in \mathcal{D}_{\text{poly}, X}^1$. Moreover, since the Lie bracket of $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly}, X}$ is Ω_X -bilinear, we may assume that $a = 1$, i.e. $\beta = D$. Now the assertion is clear from the definition of the Gerstenhaber Lie bracket; see [1, Section 3.4.2].

(2) Applying the same reduction as above, but with $D \in \mathcal{D}_{\text{poly}, X}^r$ and $r \in \{0, -1\}$, we get $[[D, f], g] \in \mathcal{D}_{\text{poly}, X}^{r-2} = 0$.

(3) By part (2) it suffices to show that $[[\beta, f], g] = 0$ for $\beta := d_{\mathcal{D}}(\gamma)$ and $\gamma \in \text{Mix}_U^0(\mathcal{D}_{\text{poly}, X}^0)$. As explained above we may further assume that $\gamma = D \in \mathcal{D}_{\text{poly}, X}^0$. Now the formulas for $d_{\mathcal{D}}$ and $[-, -]$ in [1, Section 3.4.2] imply that $[[d_{\mathcal{D}}(D), f], g] = 0$. \square

Proof of Theorem 1.14. Step 1. Take an open covering U as in property (iv). Since the sheaves $\mathcal{D}_{\text{poly}, X}^{\text{nor}, p}$ are quasi-coherent left \mathcal{D}_X -modules, it follows that $H^q(X, \mathcal{D}_{\text{poly}, X}^{\text{nor}, p}) = 0$ for all p and all $q > 0$. Therefore $\Gamma(X, \mathcal{D}_{\text{poly}, X}^{\text{nor}}) = R\Gamma(X, \mathcal{D}_{\text{poly}, X}^{\text{nor}})$ in the derived category $\mathbf{D}(\text{Mod } \mathbb{K})$. Now by [3, Theorem 3.12] the inclusion $\mathcal{D}_{\text{poly}, X}^{\text{nor}} \rightarrow \mathcal{D}_{\text{poly}, X}$ is a quasi-isomorphism, and by [3, Theorem 6.2(1)] the inclusion $\mathcal{D}_{\text{poly}, X} \rightarrow \text{Mix}_U(\mathcal{D}_{\text{poly}, X})$ is a quasi-isomorphism. According to [3, Theorem 6.2(2)] we have $\Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X})) = R\Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X}))$. The conclusion is that

$$\mathcal{D}_{\text{poly}}^{\text{nor}}(X) = \Gamma(X, \mathcal{D}_{\text{poly}, X}^{\text{nor}}) \rightarrow \Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X})) \quad (1.17)$$

is a quasi-isomorphism of complexes of \mathbb{K} -modules. But in view of [3, Proposition 6.3], this is in fact a quasi-isomorphism of DG Lie algebras.

From (1.17) we deduce that

$$\eta_{\mathcal{D}} : \mathcal{D}_{\text{poly}}^{\text{nor}}(X)[[\hbar]]^+ \rightarrow \Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X}))[[\hbar]]^+$$

is a quasi-isomorphism of DG Lie algebras. Using [3, Corollary 3.10] we see that $\text{MC}(\eta_{\mathcal{D}})$ is bijective. Therefore the diagram in property (iv) defines Q uniquely.

According to Corollary 1.13, the arrow marked $\text{MC}(\Psi_{\sigma})$ is a bijection. So we have established property (iv), except for the independence of the open covering.

Step 2. The left vertical arrow comes from the DG Lie algebra homomorphism

$$\eta_{\mathcal{T}} : \mathcal{T}_{\text{poly}}(X)[[\hbar]]^+ \rightarrow \Gamma(X, \text{Mix}_U(\mathcal{T}_{\text{poly}, X}))[[\hbar]]^+,$$

which is a quasi-isomorphism when $H^q(X, \mathcal{T}_{\text{poly}, X}^p) = 0$ for all p and all $q > 0$. So in case X is affine, the quantization map Q is bijective. This establishes property (iii).

Step 3. Now suppose $U' = \{U'_0, \dots, U'_{m'}\}$ is another such affine open covering of X , with sections $\sigma'_i : U'_i \rightarrow \text{LCC } X$. Without loss of generality we may assume that $m' \geq m$, and that $U'_i = U_i$ and $\sigma'_i = \sigma_i$ for all $i \leq m$. There is a morphism of simplicial schemes $f : U \rightarrow U'$, that is an open and closed embedding. Correspondingly there is a commutative diagram

$$\begin{array}{ccc} \text{MC}(\mathcal{T}_{\text{poly}}(X)[[\hbar]]^+) & \xrightarrow{Q} & \text{MC}(\mathcal{D}_{\text{poly}}^{\text{nor}}(X)[[\hbar]]^+) \\ \text{MC}(\eta_{\mathcal{T}}) \downarrow & & \text{MC}(\eta_{\mathcal{D}}) \downarrow \\ \text{MC}(\Gamma(X, \text{Mix}_{U'}(\mathcal{T}_{\text{poly}, X}))[[\hbar]]^+) & \xrightarrow{\text{MC}(\Psi_{\sigma'})} & \text{MC}(\Gamma(X, \text{Mix}_{U'}(\mathcal{D}_{\text{poly}, X}))[[\hbar]]^+) \\ \text{MC}(f^*) \downarrow & & \text{MC}(f^*) \downarrow \\ \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{T}_{\text{poly}, X}))[[\hbar]]^+) & \xrightarrow{\text{MC}(\Psi_{\sigma})} & \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X}))[[\hbar]]^+), \end{array}$$

where the vertical arrows on the right are bijections. We conclude that Q is independent of U and σ . This concludes the proof of property (iv).

Step 4. Suppose $f: X' \rightarrow X$ is an étale morphism. Then we can choose an affine open covering U' of X' that refines U in the obvious sense. Each of the open sets U'_i inherits an étale coordinate system, and hence a section $\sigma'_i: U'_i \rightarrow \text{LCC } X'$. We get a commutative diagram

$$\begin{array}{ccc} \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{T}_{\text{poly}, X}))[[\hbar]]^+) & \xrightarrow{\text{MC}(\psi_\sigma)} & \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X}))[[\hbar]]^+) \\ \text{VMC}(f^*) \downarrow & & \text{MC}(f^*) \downarrow \\ \text{MC}(\Gamma(X', \text{Mix}_{U'}(\mathcal{T}_{\text{poly}, X'}))[[\hbar]]^+) & \xrightarrow{\text{MC}(\psi_{\sigma'})} & \text{MC}(\Gamma(X', \text{Mix}_{U'}(\mathcal{D}_{\text{poly}, X'}))[[\hbar]]^+). \end{array}$$

This proves property (ii).

Step 5. Finally we must show that Q preserves first order terms, i.e., property (i). Let

$$\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j \in \mathcal{T}_{\text{poly}}(X)^1[[\hbar]]^+$$

be an MC solution, and let

$$\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j \in \mathcal{D}_{\text{poly}}^{\text{nor}}(X)^1[[\hbar]]^+$$

be an MC solution such that $\beta = Q(\alpha)$ modulo gauge equivalence. This means that there exists some

$$\gamma = \sum_{k \geq 1} \gamma_k \hbar^k \in \Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly}, X}))^0[[\hbar]]^+$$

such that

$$\sum_{j \geq 1} \frac{1}{j!} \Psi_{\sigma; j}(\alpha^j) = \exp(\text{af})(\exp(\gamma))(\beta),$$

with notation as in [3, Lemma 3.2]. Cf. [3, Theorem 3.8]. In the first order term (i.e., the coefficient of \hbar^1) of this equation we have

$$\Psi_{\sigma; 1}(\alpha_1) = \beta_1 - (\text{d}_{\text{mix}} + \text{d}_{\mathcal{D}})(\gamma_1); \quad (1.18)$$

see [3, Eq. (3.3)].

In order to apply Lemma 1.16(2), we are interested in the component of $\Psi_{\sigma; 1}(\alpha_1)$ living in the summand $\text{Mix}_U^0(\mathcal{D}_{\text{poly}, X}^1)$. But this is exactly

$$\text{gr}_G^0(\Psi_{\sigma; 1})(\alpha_1) \in \text{gr}_G^0 \text{Mix}_U(\mathcal{D}_{\text{poly}, X}^1) = \text{Mix}_U^0(\mathcal{D}_{\text{poly}, X}^1).$$

Since according to Theorem 1.2 we have

$$\text{gr}_G^0(\Psi_{\sigma; 1}) = \text{gr}_G^0(\text{Mix}_U(\mathcal{U}_1)),$$

it follows that the component we are interested in is

$$\mathrm{gr}_G^0(\mathrm{Mix}_U(\mathcal{U}_1))(\alpha_1) = \mathcal{U}_1(\alpha_1).$$

Now take any two local sections $f, g \in \mathcal{O}_X$. Using Lemma 1.16 we get

$$[[\Psi_{\sigma;1}(\alpha_1), f], g] = [[\mathcal{U}_1(\alpha_1), f], g] = \mathcal{U}_1(\alpha_1)(g, f) - \mathcal{U}_1(\alpha_1)(f, g) = -2\alpha_1(f, g),$$

$$[[\beta_1, f], g] = \beta_1(g, f) - \beta_1(f, g)$$

and

$$[[(\mathrm{d}_{\mathrm{mix}} + \mathrm{d}_{\mathcal{D}})(\gamma_1), f], g] = 0.$$

Combining these equations with Eq. (1.18) we see that Eq. (1.15) indeed holds. So the proof is done. \square

Corollary 1.19. *Let X be an irreducible smooth separated \mathbb{K} -scheme. Assume X is \mathcal{D} -affine. Then the quantization map Q of Theorem 1.14 may be interpreted as a canonical function*

$$Q: \frac{\{\text{formal Poisson structures on } X\}}{\text{gauge equivalence}} \rightarrow \frac{\{\text{deformation quantizations of } \mathcal{O}_X\}}{\text{gauge equivalence}}.$$

The quantization map Q preserves first order terms, and commutes with étale morphisms $f: X' \rightarrow X$. If X is affine then Q is bijective.

Proof. By definition the left side is $\mathrm{MC}(\mathcal{T}_{\mathrm{poly}}(X)[[\hbar]]^+)$. On the other hand, according to [3, Theorem 1.13] every deformation quantization of \mathcal{O}_X can be trivialized globally, and by [3, Proposition 1.14] any gauge equivalence between globally trivialized deformation quantizations is a global gauge equivalence. Hence the right side is $\mathrm{MC}(\mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X)[[\hbar]]^+)$. \square

2. Miscellaneous errors

Here is a list of minor errors in the paper [3].

(1) Section 3, bottom of page 395: the formula should be

$$\mathrm{af}(\gamma)(\omega) := [\gamma, \omega] - \mathrm{d}(\gamma) = \mathrm{ad}(\gamma)(\omega) - \mathrm{d}(\gamma) \in \mathfrak{m} \hat{\otimes} \mathfrak{g}^1,$$

(2) Definition 5.2, page 411: the formula should be

$$\nabla_{\mathcal{P}}: \mathcal{P}_X \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{P}_X.$$

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